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Lattice regularization of chiral gauge theories to all orders of perturbation theory

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ABSTRACT: In the framework of perturbation theory, it is possible to put chiral gauge theories on the lattice without violating the gauge symmetry or other fundamental principles, provided the fermion representation of the gauge group is anomaly-free. The basic elements of this construction (which starts from the Ginsparg-Wilson relation) are briefly recalled and the exact cancellation of the gauge anomaly, at any fixed value of the lattice spacing and for any compact gauge group, is then proved rigorously through a recursive procedure.

KEYWORDS: Lattice Gauge Field Theories, Gauge Symmetry, Anomalies in Field and String Theories, Renormalization Regularization and Renormalons.

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Contents

1. Introduction

In chiral gauge theories the perturbation expansion is not easy to set up consistently, because the widely used regularization methods (and also the BPHZ finite-part prescription) violate the gauge symmetry. Non-invariant counterterms must then be included in the action, with coefficients chosen so as to restore the symmetry after renormalization and removal of the regularization [1, 2, 3]. As a consequence the proof of the renormalizability of these theories is far more complicated than in the case of ordinary gauge theories. The complexity of the subtraction procedure also presents a difficulty in practice when calculating higher-order corrections to electroweak processes (see refs. [4, 5] for a recent discussion and further references).

Essentially the same (regularization plus subtraction) strategy can be adopted in lattice gauge theory, where it is referred to as the "Rome approach" [6]–[12]. In this case the BRS invariance is broken by the Wilson term, which is needed in the fermion action to avoid the infamous species-doubling problem. The symmetry is then recovered in the continuum limit after adding the appropriate counterterms.

In the present paper a different approach is described, in which the regularization preserves the gauge invariance of the theory to all orders of the gauge coupling. For many years this seemed to be excluded, but after the rediscovery of the Ginsparg-Wilson relation [13]–[20] the situation changed and a general formulation of chiral gauge theories on the lattice has emerged, where the cancellation of the symmetry-breaking terms *at any fixed value of the lattice spacing* reduces to a local cohomology problem [21]–[24]. The latter appears at the one-loop level, and once the symmetry is restored to this order of the loop expansion, the theory is guaranteed to be gauge-invariant at all higher orders too.

The cohomology problem was first solved for abelian gauge groups [25, 26], and the general solution, to all orders of the gauge coupling and for any compact gauge group, has recently been obtained by Suzuki [27].[1] In his paper Suzuki starts from the Wess-Zumino consistency condition and roughly follows the established strategies in the continuum theory [29]–[37]. This turns out to be rather complicated, but as will be shown here the exact anomaly cancellation can also be proved in a more direct and significantly simpler way.

In the next section the formulation of chiral lattice gauge theories along the lines of refs. [21]–[24] is briefly recalled. The Feynman rules in these theories (section 3) are essentially as in lattice QCD, except for the chiral projectors in the fermion propagator and a set of additional local gauge field vertices with five or more legs that constitute the solution of the cohomology problem alluded to above. In section 4

[1] Beyond perturbation theory the solution of the cohomology problem is known for abelian gauge groups [25, 26] and for $\mathrm{SU}(2)\times\mathrm{U}(1)$ [28]. The consistent formulation of chiral lattice gauge theories at this level also requires a proof of the absence of global topological obstructions. So far this has only been achieved in the abelian case [21].

an algebraic proof of the existence of this solution is given, using the classification theorem for topological fields in abelian lattice gauge theories of refs. [25, 26]. The paper ends with a short discussion of further results and a few concluding remarks.

2. Chiral lattice gauge theories

The chiral gauge theories discussed in this and the following two sections involve a multiplet of left-handed fermions but no Higgs fields or other matter fields, since these can be easily included later if so desired. For simplicity any details of the lattice formulation that are only relevant at the non-perturbative level (global anomalies, for example [38, 39, 40]) will be skipped over without further notice.

2.1 Fields and lattice action

As usual the theory is set up on a four-dimensional euclidean lattice with spacing a. The gauge group G is assumed to be a compact connected Lie group, and the gauge field is represented by link variables $U(x,\mu) \in G$, where x runs over all lattice points and $\mu = 0,\ldots,3$ labels the lattice axes. We first consider lattice Dirac fields $\psi(x)$ that transform according to some unitary representation R of the gauge group and defer the discussion of how to eliminate the right-handed components to the next subsection.

A key element of the present approach to chiral lattice gauge theories is the choice of a lattice Dirac operator D that satisfies the Ginsparg-Wilson relation [13]

$$\gamma_5\, D + D\, \gamma_5 = a\, D\, \gamma_5\, D \tag{2.1}$$

and the hermiticity condition $D^\dagger = \gamma_5 D \gamma_5$. The operator should also be local, gauge-covariant and have a number of further properties [18, 21], as any other acceptable lattice Dirac operator. A relatively simple expression, which fulfils all these requirements, is given by [16]

$$D = \frac{1}{a}\left\{1 - A\left(A^\dagger A\right)^{-1/2}\right\}, \quad A = 1 - a\, D_{\rm w}\,, \tag{2.2}$$

where $D_{\rm w}$ denotes the standard Wilson-Dirac operator

$$D_{\rm w} = \frac{1}{2}\left\{\gamma_\mu\left(\nabla_\mu^* + \nabla_\mu\right) - a\,\nabla_\mu^*\nabla_\mu\right\} \tag{2.3}$$

(see appendix A for unexplained notations). In the following we shall stick to this operator, but it should be emphasized that other acceptable solutions of the Ginsparg-Wilson relation would do just as well. For perturbation theory it would actually be sufficient to provide a solution in the form of a formal power series expansion in the gauge potential.

In the present context the standard plaquette action

$$S_{\rm G}[U] = \frac{1}{g_0^2}\sum_x \sum_{\mu,\nu} {\rm Re\,tr}\,\{1 - P(x,\mu,\nu)\}\,, \tag{2.4}$$

$$P(x,\mu,\nu) = U(x,\mu)\,U(x+a\hat{\mu},\nu)\,U(x+a\hat{\nu},\mu)^{-1}\,U(x,\nu)^{-1}\,, \tag{2.5}$$

is a possible choice for the gauge field action, with g_0 the bare coupling, and the fermion action is taken to be of the usual form

$$S_{\rm F}[U,\overline{\psi},\psi] = a^4\sum_x \overline{\psi}(x)D\psi(x)\,. \tag{2.6}$$

At this stage the theory thus looks like lattice QCD, except for the fact that we have allowed the fermions to be in an arbitrary representation R of the gauge group.

2.2 Chiral projection

An important consequence of the Ginsparg-Wilson relation is that the fermion action admits an exact chiral symmetry [17] that can be used to separate the chiral components of the fermion field in a natural way [19, 20, 21]. One first observes that the operator $\hat{\gamma}_5 = \gamma_5(1-aD)$ satisfies

$$(\hat{\gamma}_5)^\dagger = \hat{\gamma}_5\,, \qquad (\hat{\gamma}_5)^2 = 1\,, \qquad D\hat{\gamma}_5 = -\gamma_5 D\,. \tag{2.7}$$

The fermion action thus splits into left- and right-handed parts if the chiral projectors for fermion and antifermion fields are defined through

$$\hat{P}_\pm = \frac{1}{2}\left(1 \pm \hat{\gamma}_5\right), \qquad P_\pm = \frac{1}{2}\left(1 \pm \gamma_5\right), \tag{2.8}$$

respectively. In particular, by imposing the constraints

$$\hat{P}_-\psi = \psi\,, \qquad \overline{\psi}P_+ = \overline{\psi}\,, \tag{2.9}$$

the right-handed components are eliminated and one obtains a chiral gauge theory that is completely consistent at the classical level.

2.3 Correlation functions

Expectation values of products $\mathcal{O}$ of the basic fields are defined through the functional integral

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int {\rm D}[U]\,{\rm D}[\psi]\,{\rm D}[\overline{\psi}]\,\mathcal{O}\,e^{-S_{\rm G}[U]-S_{\rm F}[U,\overline{\psi},\psi]}\,, \tag{2.10}$$

where ${\rm D}[U]$ denotes the standard integration measure for lattice gauge fields and the normalization factor $\mathcal{Z}$ is defined through the requirement that $\langle 1\rangle = 1$. The fermion integral should be restricted to the subspace of left-handed fields, which can be easily done for any given gauge field configuration. However, since the subspace of left-handed fields changes with the gauge field, there is no obvious way to fix the relative phase of the fermion integration measure at different points in field space. As a consequence the fermion partition function

$$e^{-S_{\rm eff}[U]} = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\, e^{-S_{\rm F}[U,\overline{\psi},\psi]} \tag{2.11}$$

has a non-trivial phase ambiguity [21]–[24].

Apart from this, the fermion integral is well-defined and of the gaussian type. Equation (2.10) can thus be rewritten in the form

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}[U]\,\{\mathcal{O}\}_{\rm F}\, e^{-S_{\rm G}[U]-S_{\rm eff}[U]}\,, \tag{2.12}$$

where $\{\mathcal{O}\}_{\rm F}$ is a sum of Wick contractions that are obtained by applying Wick's theorem to the fermion fields in $\mathcal{O}$ and substituting

$$\left\{\psi(x)\overline{\psi}(y)\right\}_{\rm F} = \hat{P}_{-}S(x,y)P_{+}\,, \qquad DS(x,y) = a^{-4}\delta_{xy}\,, \tag{2.13}$$

for the basic contraction. Note the presence of the chiral projectors in this formula, which make it explicit that the propagating fermions are left-handed.

2.4 Measure term and gauge invariance

To complete the definition of the lattice theory, we now need to say how the phase of the fermion measure is to be determined. Since only the relative phase at different points in field space matters, the problem may be approached by computing the change of the effective action $S_{\rm eff}$ under variations

$$\delta_{\eta}U(x,\mu) = a\eta_{\mu}(x)U(x,\mu)\,, \qquad \eta_{\mu}(x) = \eta_{\mu}^{a}(x)T^{a}\,, \tag{2.14}$$

of the link field (cf. appendix A). Apart from the naively expected term, the result of this calculation [21]–[24]

$$\delta_{\eta}S_{\rm eff} = -\,\mathrm{Tr}\left\{\delta_{\eta}D\hat{P}_{-}D^{-1}P_{+}\right\} + i\mathfrak{L}_{\eta} \tag{2.15}$$

involves a second term (the *measure term*) that arises from the implicit dependence of the fermion measure on the gauge field. $\mathfrak{L}_{\eta}$ is linear in the field variation,

$$\mathfrak{L}_{\eta} = a^{4}\sum_{x}\eta_{\mu}^{a}(x)\,j_{\mu}^{a}(x)\,, \tag{2.16}$$

where $j_{\mu}(x)$ is a function of the gauge field that contains all the non-trivial information about the phase of the fermion measure.

At this point little is known about this current, but it is straightforward to write down a few general requirements that turn out to be very restrictive and essentially fix the phase of the measure [21]–[24]. In perturbation theory the situation is particularly simple, and the only conditions that must be fulfilled are[2]

(a) The current $j_\mu(x)$ is a gauge-covariant local field.

(b) $\mathfrak{L}_\eta$ satisfies the integrability condition

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + a\mathfrak{L}_{[\eta,\zeta]} = i\,\mathrm{Tr}\left\{\hat{P}_- \left[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-\right]\right\} \tag{2.17}$$

for all field variations $\eta_\mu(x)$ and $\zeta_\mu(x)$ that do not depend on the gauge field.

The measure term $\mathfrak{L}_\eta$ may then be interpreted as a local counterterm, which has to be included to ensure the integrability of the right-hand side of eq. (2.15). Since the existence of an underlying fermion measure is guaranteed if (b) holds [23], one can in fact *define the theory* through eqs. (2.12)–(2.16), with some particular choice of the current $j_\mu(x)$ that satisfies conditions (a) and (b).

In the following we adopt this point of view and shall show (in section 4) that such a current can be constructed to all orders of the gauge coupling if the fermion multiplet is anomaly-free, i.e. if the tensor

$$d_R^{abc} = 2i\,\mathrm{tr}\left\{R(T^a)\left[R\left(T^b\right)R\left(T^c\right) + R\left(T^c\right)R\left(T^b\right)\right]\right\} \tag{2.18}$$

vanishes. There are no further restrictions on the fermion representation or the gauge group, and the solution is unique up to irrelevant local terms that amount to a redefinition of the lattice action of the gauge field.

Once the phase ambiguity has been fixed in this way, the effective action S_{eff} (and thus the whole theory) can be shown to be gauge-invariant. The proof of this important result is given in appendix B. Here we only note that infinitesimal gauge transformations are generated by lattice fields $\omega(x)$ with values in the Lie algebra of G. The corresponding variations of the link variables are obtained by substituting

$$\eta_\mu(x) = -\nabla_\mu\,\omega(x) \tag{2.19}$$

in eq. (2.14), and the gauge invariance of the effective action is then equivalent to the statement that $\delta_\eta S_{\mathrm{eff}} = 0$ for all these variations.

3. Perturbation theory

From the point of view of perturbation theory, the theories defined above are rather similar to lattice QCD with Wilson fermions. Important differences result from the use of a relatively complicated lattice Dirac operator and from the presence of the measure term, which gives rise to additional gauge field vertices. In this section we mainly address these issues, while for the more common aspects of lattice perturbation theory the reader is referred to refs. [41]–[44], for example.

[2]The notion of locality which is being used here is the same as in the earlier work on the subject (see ref. [25], for example). What precisely this means in perturbation theory is explained in section 3.

3.1 Gauge fixing and BRS symmetry

When the gauge coupling g_0 is taken to zero, the functional integral (2.12) is dominated by the field configurations in the vicinity of the gauge orbit that passes through the trivial field $U(x,\mu) = 1$. The perturbation expansion essentially amounts to a saddle-point expansion about this orbit. As usual the gauge degrees of freedom are first eliminated by including a gauge-fixing term, and the gauge-fixed theory then has an exact BRS symmetry, for any value of the lattice spacing [41, 42]. Note that the effective action $S_{\rm eff}$ does not interfere with this, since it is gauge-invariant and of second order in the gauge coupling.

In the gauge-fixed theory the gauge field is parametrized through

$$U(x,\mu) = e^{g_0 a A_\mu(x)}, \qquad A_\mu(x) = A_\mu^a(x)\, T^a. \tag{3.1}$$

The integration variables are then the components $A_\mu^a(x)$ of the gauge potential, and the perturbation expansion is obtained straightforwardly by expanding all entries in the functional integral in powers of g_0. Apart from the terms that derive from the effective action $S_{\rm eff}$ or the fermion propagators in the Wick contracted product $\{\mathcal{O}\}_{\rm F}$, the resulting Feynman rules are exactly as in the pure gauge theory and will not be discussed here.

3.2 Expansion of the fermion propagator

The fermion propagator (2.13) may be written in the form

$$\left\{\psi(x)\,\overline{\psi}(y)\right\}_{\rm F} = S(x,y)P_+ \tag{3.2}$$

and our task is thus to expand the Green function $S(x,y)$ in powers of the gauge coupling. This has previously been described in refs. [45]–[49], but it may be worth while to briefly go through the main steps of this calculation to elucidate the structure of the free propagator and of the fermion-gauge-field vertices.

In position space the Dirac operator D is represented by a kernel $D(x,y)$ through

$$D\psi(x) = a^4 \sum_y D(x,y)\,\psi(y). \tag{3.3}$$

It suffices to work out the perturbation expansion

$$D(x,y) = \sum_{k=0}^{\infty} D^{(k)}(x,y), \tag{3.4}$$

$$D^{(k)}(x,y) = \frac{g_0^k}{k!}\, a^{4k} \sum_{z_1,\ldots,z_k} D^{(k)}(x,y,z_1,\ldots,z_k)^{a_1\cdots a_k}_{\mu_1\cdots\mu_k} A^{a_1}_{\mu_1}(z_1)\cdots A^{a_k}_{\mu_k}(z_k), \tag{3.5}$$

of the Dirac operator, since the Green function is then obtained as usual through the Neumann series

$$S(x,y) = S^{(0)}(x,y) - a^8 \sum_{u,v} S^{(0)}(x,u)D^{(1)}(u,v)S^{(0)}(v,y) + \cdots. \tag{3.6}$$

Note that the kernels on the right-hand side of eq. (3.5) are just the bare vertices of the theory in position space with two fermion and k gauge field legs.

Starting from the definition (2.2) of the Dirac operator, it is possible to compute these kernels analytically in momentum space. Beyond the lowest orders the calculation leads to increasingly complicated expressions, but it is in principle straightforward and programmable. An important simplification derives from the fact that the operator $A^{\dagger}A$ does not act on the Dirac indices at $g_0 = 0$. It is then not difficult to show that the free Dirac operator is given by

$$aD^{(0)}(x,y) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)} \left\{1 - \left(1 - \frac{1}{2}\,a^2\hat{p}^2 - ia\gamma_\mu \mathring{p}_\mu\right)\lambda(p)^{-1/2}\right\}, \quad (3.7)$$

$$\lambda(p) = 1 + \frac{1}{2}\,a^4 \sum_{\mu<\nu} \hat{p}_\mu^2\,\hat{p}_\nu^2\,, \quad (3.8)$$

where the standard notations $\hat{p}_\mu = (2/a)\sin(ap_\mu/2)$ and $\mathring{p}_\mu = (1/a)\sin(ap_\mu)$ have been used.

To compute the fermion-gauge-field vertices, we expand the integrand in the integral representation

$$aD = 1 - \int_{-\infty}^{\infty} \frac{dt}{\pi}\, A\left(t^2 + A^{\dagger}A\right)^{-1} \quad (3.9)$$

in powers of the gauge coupling [45]. This yields a sum of products of simple operators, and after passing to momentum space the vertices are obtained in the form of integrals of the type

$$\int_{-\infty}^{\infty} \frac{dt}{\pi}\, \frac{P(q_1,\ldots,q_l)}{(t^2+\lambda(p_1))\cdots(t^2+\lambda(p_r))}\,, \quad (3.10)$$

where the numerator is a polynomial in the sines and cosines of the incoming momenta and $p_1,\ldots,p_r$ are integer linear combinations of these. The integrals may finally be evaluated using the residue theorem.

It should be obvious from the above that the vertices are analytic functions of the incoming momenta in a complex region around the Brillouin zone. The kernel $D^{(k)}(x,y,z_1,\ldots,z_k)^{a_1\ldots a_k}_{\mu_1\ldots\mu_k}$ consequently falls off exponentially when the distance between any two of its arguments becomes large. Moreover the characteristic decay length is a fixed number in lattice units and hence microscopically small from the point of view of the continuum limit.

The free lattice Dirac operator and the fermion-gauge-field vertices are thus local, as should be the case in a well-behaved theory. In view of the general results of ref. [18], this does not come as a surprise, but having made it explicit what locality means in the present context will be helpful in the following.

JHEP06(2000)028

3.3 Expansion of the effective action

In the perturbation expansion of the functional integral (2.12), the effective action $S_{\rm eff}$ gives rise to additional (non-local) gauge field vertices at the one-loop level. Up to an additive constant, the expansion of $S_{\rm eff}$ in powers of g_0 reads

$$S_{\rm eff}=\sum_{k=2}^{\infty}\frac{g_0^k}{k!}a^{4k}\sum_{z_1,\ldots,z_k}V^{(k)}(z_1,\ldots,z_k)^{a_1\cdots a_k}_{\mu_1\cdots\mu_k}A^{a_1}_{\mu_1}(z_1)\cdots A^{a_k}_{\mu_k}(z_k)\,. \tag{3.11}$$

The vertices $V^{(k)}$ can be computed by differentiating eq. (2.15) with respect to the gauge field, but instead of δ_η another differential operator $\bar{\delta}_\eta$ should better be used for this, which acts on the gauge potential according to

$$\bar{\delta}_\eta A_\mu(x)=\eta_\mu(x) \tag{3.12}$$

(cf. appendix A). The kth order vertex is then obtained by applying this operator k times to the effective action and setting the gauge potential to zero at the end of the calculation.

In terms of $\bar{\delta}_\eta$, eq. (2.15) assumes the form

$$\bar{\delta}_\eta S_{\rm eff}=-\,{\rm Tr}\left\{\bar{\delta}_\eta D\hat{P}_-D^{-1}P_+\right\}+ig_0\mathfrak{L}_{\bar{\eta}}\,, \tag{3.13}$$

$$\bar{\eta}_\mu(x)=\left\{1+\frac{1}{2}\,g_0a\,{\rm Ad}\,A_\mu(x)+\cdots\right\}\eta_\mu(x)\,, \tag{3.14}$$

where the higher-order terms in the curly bracket are given explicitly in appendix A. The differentiation of the trace term,

$$-\underbrace{\bar{\delta}_\eta\cdots\bar{\delta}_\eta}_{k-1\ \text{times}}{\rm Tr}\left\{\bar{\delta}_\eta D\hat{P}_-D^{-1}P_+\right\}\Big|_{A_\mu=0}=g_0^ka^{4k}\sum_{z_1,\ldots,z_k}V_{\rm F}^{(k)}(z_1,\ldots,z_k)^{a_1\ldots a_k}_{\mu_1\ldots\mu_k}\times \\ \times\,\eta^{a_1}_{\mu_1}(z_1)\cdots\eta^{a_k}_{\mu_k}(z_k)\,, \tag{3.15}$$

yields the fermion loop contribution to the vertices $V^{(k)}$. As in the case of the fermion propagator, the projector $\hat{P}_-$ on the left-hand side of this equation may be omitted. The derivatives then act on the Dirac operator or its inverse, and as a result all fermion one-loop diagrams with k external gauge field lines are generated.

In the next section the measure term will be obtained in the form of a power series

$$\mathfrak{L}_\eta=\sum_{k=4}^{\infty}\frac{g_0^k}{k!}a^{4k+4}\sum_{x,\ldots,z_k}L^{(k)}(x,z_1,\ldots,z_k)^{aa_1\ldots a_k}_{\mu\mu_1\ldots\mu_k}\times \\ \times\,\eta^a_\mu(x)A^{a_1}_{\mu_1}(z_1)\cdots A^{a_k}_{\mu_k}(z_k)\,, \tag{3.16}$$

with coefficients $L^{(k)}$ that are translation-invariant and local (with exponentially decaying tails as in the case of the fermion-gauge-field vertices discussed above). Since the series starts at $k=4$, the vertices that derive from the measure term,

$$ig_0\underbrace{\bar{\delta}_\eta\cdots\bar{\delta}_\eta}_{k-1\ \text{times}}\mathfrak{L}_{\bar{\eta}}\big|_{A_\mu=0}=g_0^ka^{4k}\sum_{z_1,\ldots,z_k}V_{\rm M}^{(k)}(z_1,\ldots,z_k)^{a_1\ldots a_k}_{\mu_1\ldots\mu_k}\eta^{a_1}_{\mu_1}(z_1)\cdots\eta^{a_k}_{\mu_k}(z_k)\,, \tag{3.17}$$

only occur at the fifth and higher orders of the gauge coupling. They are local linear combinations of the coefficients in eq. (3.16), properly symmetrized so as to comply with Bose symmetry.

4. Determination of the measure term

We are now left with the task of determining the coefficients in eq. (3.16) such that conditions (a) and (b) are satisfied to all orders of the gauge coupling (cf. section 2). As explained in ref. [23], this is equivalent to solving a local cohomology problem in 4+2 dimensions, but we shall not make use of this connection here and instead construct the solution directly through a recursive procedure, assuming that the fermion representation of the gauge group is anomaly-free. The "curvature"

$$\mathfrak{F}_{\eta\zeta} = i\,\mathrm{Tr}\left\{\hat{P}_{-}\left[\delta_{\eta}\hat{P}_{-},\delta_{\zeta}\hat{P}_{-}\right]\right\}, \tag{4.1}$$

which appears on the right-hand side of the integrability condition (2.17), plays an important role in this construction, and its properties are thus worked out first.

4.1 Gauge invariance, charge conjugation and the Bianchi identity

From the definition of the differential operator δ_η and the gauge-covariance of the projector to the left-handed fields, it follows that $\mathfrak{F}_{\eta\zeta}$ is invariant under gauge transformations if $\eta_\mu(x)$ and $\zeta_\mu(x)$ are transformed like gauge-covariant local fields.

The lattice Dirac operator D has the same charge conjugation properties as the Dirac operator in the continuum theory. In terms of the kernel $D(x,y)$, this means that its complex conjugate is given by

$$D(x,y)^{*} = BD(x,y)_{R\rightarrow R^{*}}B^{-1}, \tag{4.2}$$

where B is a 4×4 matrix such that $B\gamma_\mu B^{-1}=\gamma_\mu^{*}$. The kernel of the projector to the left-handed fields transforms exactly in the same way, and since the differential operator δ_η is real, it follows that

$$\mathfrak{F}_{\eta\zeta} = (\mathfrak{F}_{\eta\zeta})^{*} = -(\mathfrak{F}_{\eta\zeta})_{R\rightarrow R^{*}}. \tag{4.3}$$

In particular, the curvature vanishes if the representation R is real or pseudo-real.

Apart from being antisymmetric, $\mathfrak{F}_{\eta\zeta}$ also satisfies the Bianchi identity

$$\delta_{\eta}\mathfrak{F}_{\zeta\lambda}+\delta_{\zeta}\mathfrak{F}_{\lambda\eta}+\delta_{\lambda}\mathfrak{F}_{\eta\zeta}+a\mathfrak{F}_{[\eta,\zeta]\lambda}+a\mathfrak{F}_{[\zeta,\lambda]\eta}+a\mathfrak{F}_{[\lambda,\eta]\zeta}=0. \tag{4.4}$$

This can be proved in a few lines, using the commutator rule (A.6) and $(\hat{\gamma}_5)^2=1$, which implies the vanishing of $\mathrm{Tr}\{(\delta_\eta\hat{\gamma}_5)(\delta_\zeta\hat{\gamma}_5)(\delta_\lambda\hat{\gamma}_5)\}$.

4.2 Expansion of the curvature in powers of g_0

When the perturbation expansion of the Dirac operator is inserted on the right-hand side of eq. (4.1), a series of the form

$$\mathfrak{F}_{\eta\zeta} = \sum_{k=0}^{\infty} \frac{g_0^k}{k!} a^{4k+8} \sum_{x,\ldots,z_k} F^{(k)}(x,y,z_1,\ldots,z_k)^{aba_1\ldots a_k}_{\mu\nu\mu_1\ldots\mu_k} \times \\ \times \eta^a_\mu(x)\,\zeta^b_\nu(y) A^{a_1}_{\mu_1}(z_1)\cdots A^{a_k}_{\mu_k}(z_k) \qquad (4.5)$$

is obtained, with local coefficients $F^{(k)}$ that are sums of products of the fermion-gauge-field vertices. They are invariant under the adjoint action of G (i.e. under constant gauge transformations) and change sign when the representation R is replaced by R^*.

We now show that *all terms of order $k \leq 2$ are equal to zero* as a consequence of the anomaly cancellation condition $d_R^{abc} = 0$. The proof is simple and starts with the observation that the link variables in the Dirac operator only appear in the representation R. Taking the charge conjugation symmetry into account, it follows from this and the general structure of the fermion-gauge-field vertices that $F^{(k)}$ must be a linear combination of the tensors

$$\mathrm{tr}\,\{R(T^{c_1})\cdots R(T^{c_{k+2}})\} + (-1)^{k+1}\,\mathrm{tr}\,\{R(T^{c_{k+2}})\cdots R(T^{c_1})\}\,, \qquad (4.6)$$

where $c_1,\ldots,c_{k+2}$ is any permutation of the indices $a,b,a_1,\ldots,a_k$. In particular, the leading order term $F^{(0)}$ is equal to zero, and the same is true for $F^{(1)}$, because the tensor (4.6) is proportional to $d_R^{c_1c_2c_3}$ in this case.

For $k=2$ the vanishing of the tensor can be proved by inverting the order of the generators in both traces simultaneously, using

$$\left[R\left(T^a\right), R\left(T^b\right)\right] = f^{abc} R(T^c)\,. \qquad (4.7)$$

The commutator terms that are generated in this way do not contribute, since they are proportional to the d_R^{abc} symbol. At the end of the calculation, the tensor is thus reproduced with the opposite sign, which is only possible if it is equal to zero.

4.3 Solution of the integrability condition to lowest order

On the left-hand side of the integrability condition (2.17), the differential operators decrease the order of each term in the series (3.16) since

$$\delta_\eta = g_0^{-1}\bar{\delta}_\eta + \mathrm{O}(1) \qquad (4.8)$$

(cf. appendix A). The first possibly non-zero term is thus of order g_0^3 and if we define the lowest-order parts of the measure term and the curvature through

$$\check{\mathfrak{L}}_\eta = \frac{1}{4!}\frac{\partial^4}{\partial g_0^4}\,\mathfrak{L}_\eta\Bigg|_{g_0=0}, \qquad \check{\mathfrak{F}}_{\eta\zeta} = \frac{1}{3!}\frac{\partial^3}{\partial g_0^3}\,\mathfrak{F}_{\eta\zeta}\Bigg|_{g_0=0}, \qquad (4.9)$$

the integrability condition at this order of the gauge coupling becomes

$$\bar{\delta}_\eta \check{\mathfrak{L}}_\zeta - \bar{\delta}_\zeta \check{\mathfrak{L}}_\eta = \check{\mathfrak{F}}_{\eta\zeta} \,. \tag{4.10}$$

Condition (a) must be satisfied too, and this implies that $\check{\mathfrak{L}}_\eta$ has to be invariant under *linearized gauge transformations*

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \omega(x) \tag{4.11}$$

and also under constant gauge transformations (where both the gauge potential and $\eta_\mu(x)$ are rotated).

The following chain of arguments, which leads to a solution $\check{\mathfrak{L}}_\eta$ of the problem, only makes use of the locality, gauge invariance, homogeneity and antisymmetry of $\check{\mathfrak{F}}_{\eta\zeta}$ and of the Bianchi identity

$$\bar{\delta}_\eta \, \check{\mathfrak{F}}_{\zeta\lambda} + \bar{\delta}_\zeta \, \check{\mathfrak{F}}_{\lambda\eta} + \bar{\delta}_\lambda \, \check{\mathfrak{F}}_{\eta\zeta} = 0 \tag{4.12}$$

that derives from eq. (4.4). For clarity, the construction is broken up in four steps.

1. We first introduce a linear functional $\mathfrak{H}_\eta$ through

$$\mathfrak{H}_\eta = -\frac{1}{5}\, \check{\mathfrak{F}}_{\eta\lambda}\big|_{\lambda_\mu = A_\mu} = a^4 \sum_x \eta^a_\mu(x)\, h^a_\mu(x) \,, \tag{4.13}$$

   where the second equation defines the current $h_\mu(x)$. Using the Bianchi identity and the homogeneity of $\check{\mathfrak{F}}_{\eta\zeta}$, it is straightforward to show that

$$\bar{\delta}_\eta \mathfrak{H}_\zeta - \bar{\delta}_\zeta \mathfrak{H}_\eta = \frac{1}{5}\left\{ 2\check{\mathfrak{F}}_{\eta\zeta} - \left(\bar{\delta}_\eta \check{\mathfrak{F}}_{\zeta\lambda} + \bar{\delta}_\zeta \check{\mathfrak{F}}_{\lambda\eta}\right)_{\lambda_\mu = A_\mu} \right\} = \check{\mathfrak{F}}_{\eta\zeta} \,, \tag{4.14}$$

   and $\mathfrak{H}_\eta$ thus solves the leading-order form (4.10) of the integrability condition.

2. The current $h_\mu(x)$ itself may be gauge-dependent, but its divergence

$$q(x) = \partial^*_\mu h_\mu(x) \tag{4.15}$$

   can be proved to be invariant under linearized gauge transformations. To this end let us consider two gauge variations $\eta_\mu(x) = \partial_\mu \omega(x)$ and $\zeta_\mu(x) = \partial_\mu \sigma(x)$. Since the lowest-order part of the curvature is invariant under such variations, we have

$$\bar{\delta}_\lambda \, \check{\mathfrak{F}}_{\eta\zeta} = -\bar{\delta}_\eta \, \check{\mathfrak{F}}_{\zeta\lambda} - \bar{\delta}_\zeta \, \check{\mathfrak{F}}_{\lambda\eta} = 0 \,, \tag{4.16}$$

   which proves that $\check{\mathfrak{F}}_{\eta\zeta}$ is independent of the gauge potential and hence equal to zero. Recalling the definition (4.13), we now note that

$$\bar{\delta}_\zeta \, \mathfrak{H}_\eta = -\frac{1}{5}\, \check{\mathfrak{F}}_{\eta\zeta} = 0 \,. \tag{4.17}$$

   After substituting $\eta_\mu(x) = \partial_\mu \omega(x)$ and performing a partial summation, this is easily seen to be equivalent to the statement that the divergence (4.15) is invariant under linearized gauge transformations.

3. From the above one concludes that $q(x)$ is a *topological field*, i.e. it is local, invariant under linearized gauge transformations and satisfies

$$a^4 \sum_x \bar{\delta}_\lambda q(x) = 0 \tag{4.18}$$

for all variations $\lambda_\mu(x)$ of the gauge potential. In lattice gauge theories with gauge group U(1), it is known that any field with these properties can be written as a sum of a Chern polynomial plus a topologically trivial term equal to the divergence of a gauge-invariant local current [25, 26].

The theorem and its proof literally carry over to the present situation, where the components $A_\mu^1(x), \ldots, A_\mu^n(x)$ behave like independent abelian gauge fields. We are actually dealing with a particularly simple case, because $q(x)$ is homogeneous in the gauge potential of degree 4, while the general Chern polynomial has degree 2 in four dimensions. The classification theorem thus implies

$$q(x) = \partial_\mu^* k_\mu(x)\,, \tag{4.19}$$

where $k_\mu(x)$ is a local current that is invariant under linearized gauge transformations. In refs. [25, 26] the current has been constructed algebraically using a lattice version of the Poincaré lemma, and while the resulting expression is rather complicated, it shows that $k_\mu(x)$ may be assumed to be homogeneous of degree 4 and to transform covariantly under the adjoint action of the gauge group.

4. The lowest-order part of the measure term is now given by

$$\check{\mathfrak{L}}_\eta = \mathfrak{H}_\eta + \bar{\delta}_\eta \left\{ \frac{1}{4}\, a^4 \sum_x A_\mu^a(x) k_\mu^a(x) \right\}. \tag{4.20}$$

Since the second term has vanishing curvature, it is immediately clear from eq. (4.14) that the integrability condition in its leading-order form (4.10) is satisfied. $\check{\mathfrak{L}}_\eta$ is also local, homogeneous of degree 4 and invariant under constant gauge transformations.

To check the invariance of $\check{\mathfrak{L}}_\eta$ under linearized gauge transformations, we consider a gauge variation $\zeta_\mu(x) = \partial_\mu \sigma(x)$ and note that

$$\bar{\delta}_\zeta \check{\mathfrak{L}}_\eta = -\frac{1}{5}\, \check{\mathfrak{F}}_{\eta\zeta} + \bar{\delta}_\eta \left\{ \frac{1}{4}\, a^4 \sum_x \zeta_\mu^a(x)\, k_\mu^a(x) \right\}. \tag{4.21}$$

Use has been made here of the definition (4.13) and of the gauge invariance of $\check{\mathfrak{F}}_{\eta\lambda}$ and $k_\mu(x)$. We already know that eq. (4.10) holds, and the curvature $\check{\mathfrak{F}}_{\eta\zeta}$

may thus be eliminated using this relation. As a result one obtains

$$\begin{aligned}\bar{\delta}_\zeta \check{\mathfrak{L}}_\eta &= \bar{\delta}_\eta \left\{ -\frac{1}{4}\,\check{\mathfrak{L}}_\zeta + \frac{5}{16}\,a^4 \sum_x \zeta^a_\mu(x) k^a_\mu(x) \right\} \\ &= \bar{\delta}_\eta \left\{ \frac{1}{4}\,a^4 \sum_x \sigma^a(x) \left[\partial^*_\mu h^a_\mu(x) - \partial^*_\mu k^a_\mu(x) \right] \right\} = 0\,, \end{aligned} \qquad (4.22)$$

where the last equality follows from eqs. (4.15) and (4.19). $\check{\mathfrak{L}}_\eta$ thus fulfils all conditions to be an acceptable choice of the leading-order part of the measure term.

4.4 Determination of the higher-order terms

The clue to the construction of the measure term at the next-to-lowest order of the gauge coupling is the fact that there exists a gauge-invariant local functional $\mathfrak{L}^{(4)}_\eta$ whose lowest-order part coincides with $g_0^4 \check{\mathfrak{L}}_\eta$. There are actually many such expressions, and a particularly simple one is given in appendix C. Once this is established, a subtracted measure term and associated curvature may be defined through

$$\mathfrak{L}'_\eta = \mathfrak{L}_\eta - \mathfrak{L}^{(4)}_\eta\,, \qquad (4.23)$$

$$\mathfrak{F}'_{\eta\zeta} = \mathfrak{F}_{\eta\zeta} - \left\{ \delta_\eta \mathfrak{L}^{(4)}_\zeta - \delta_\zeta \mathfrak{L}^{(4)}_\eta + a\mathfrak{L}^{(4)}_{[\eta,\zeta]} \right\}, \qquad (4.24)$$

in terms of which the integrability condition (2.17) assumes the form

$$\delta_\eta \mathfrak{L}'_\zeta - \delta_\zeta \mathfrak{L}'_\eta + a\mathfrak{L}'_{[\eta,\zeta]} = \mathfrak{F}'_{\eta\zeta}\,. \qquad (4.25)$$

The new curvature $\mathfrak{F}'_{\eta\zeta}$ is of order g_0^4, but has otherwise the same basic properties (locality, gauge invariance, homogeneity, antisymmetry, Bianchi identity) as $\mathfrak{F}_{\eta\zeta}$. In particular, the lowest-order part of $\mathfrak{L}'_\eta$ can be determined by going through the steps in the previous subsection again, with the obvious changes that need to be made because the degree of homogeneity has increased by 1.

Evidently this procedure defines a recursion, which results in a series

$$\mathfrak{L}_\eta = \sum_{k=4}^{\infty} \mathfrak{L}^{(k)}_\eta\,, \qquad (4.26)$$

where $\mathfrak{L}^{(k)}_\eta$ is of order g_0^k. The so constructed solution has all the required properties, and by expanding the terms in eq. (4.26) in powers of the gauge coupling, one finally obtains the coefficients $L^{(k)}$.

5. Further comments and results

5.1 Lattice symmetries

The imaginary part of the effective action should transform like a pseudoscalar under lattice rotations and reflections, but with the measure term $\mathfrak{L}_\eta$ constructed in the

JHEP06(2000)028

previous section, this is not guaranteed. We can, however, enforce the symmetry by replacing $\mathfrak{L}_\eta$ through the symmetrized expression

$$\frac{1}{2^4\, 4!} \sum_{\Lambda \in \mathrm{O}(4,\mathbb{Z})} \det \Lambda\ \mathfrak{L}_\eta\big|_{U\to U^\Lambda, \eta\to\eta^\Lambda} . \tag{5.1}$$

Conditions (a) and (b) then are still fulfilled and the effective action has the desired transformation behaviour.

The average (5.1) is taken over the group of integer orthogonal matrices Λ. They act on the lattice points and the gauge field in the usual way, while in the case of the field $\eta_\mu(x)$ the transformation law is such that

$$\left[e^{ta\eta_\mu(x)} U(x,\mu)\right]^\Lambda = e^{ta\eta_\mu^\Lambda(x)} U^\Lambda(x,\mu) . \tag{5.2}$$

This implies a simple transformation behaviour of the differential operator δ_η and the statement made above can then be proved straightforwardly.

5.2 Uniqueness of the measure term

Conditions (a) and (b) do not fix the measure term uniquely, but if we require that the lattice symmetries are preserved, the difference $\Delta\mathfrak{L}_\eta$ between any two solutions can be shown to be of the form

$$\Delta\mathfrak{L}_\eta = a^4 \sum_x \delta_\eta \Omega(x) , \tag{5.3}$$

where $\Omega(x)$ is a gauge-invariant, pseudoscalar local field. Apart from the Chern monomials, which do not contribute in perturbation theory due to their topological nature, any field of this type has dimension greater than 4. Different choices of the measure term thus amount to including further terms in the lattice action that are expected to be irrelevant in the continuum limit (up to finite renormalizations).

5.3 Anomalous theories

If the fermion multiplet is anomalous, the expansion (4.5) of the curvature $\mathfrak{F}_{\eta\zeta}$ starts at $k = 1$ with a term proportional to d_R^{abc} and the lowest-order part of the measure term thus has to be a polynomial in the gauge potential of degree 2. The argumentation in subsection 4.3 then leads to a topological field $q(x)$ as before, but the field now has degree 2 and can be topologically non-trivial. From the results obtained in refs. [23, 25, 26], it is in fact possible to infer that

$$q(x) = -\frac{1}{192\pi^2} d_R^{abc} \epsilon_{\mu\nu\rho\sigma} T^a F_{\mu\nu}^b(x) F_{\rho\sigma}^c(x + a\hat{\mu} + a\hat{\nu}) + \partial_\mu^* k_\mu(x) , \tag{5.4}$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ denotes the linearized gauge field tensor. The construction of the measure term along the lines of section 4 thus breaks down at this point.

5.4 Renormalizability

The lattice theories defined in this paper provide a gauge-invariant regularization of anomaly-free chiral gauge theories to all orders of the gauge coupling. Moreover the construction preserves the lattice symmetries (to the extent that this can be expected in a chiral theory) and the propagators and basic vertices have all the required properties for the Reisz power counting theorem [50, 41] to apply.

As a consequence there is little doubt that these theories are multiplicatively renormalizable, i.e. it suffices to renormalize the gauge coupling and the fields to be able to pass to the continuum limit. Using techniques similar to those previously employed in the case of lattice QCD [51, 52], it seems in fact quite likely that a rigorous proof of the multiplicative renormalizability can be given, even though the combinatorial aspects of the renormalization procedure may have to be reconsidered, since the functional integral does not have the standard form with a local action and field-independent integration measures for all fields.

5.5 Right-handed fermions and Higgs fields

In theories with left- and right-handed fermions there are two multiplets of chiral fields, $\psi_L(x)$ and $\psi_R(x)$, that transform according to some representations R_L and R_R of the gauge group. Depending on which field the lattice Dirac operator D acts, the appropriate covariant difference operators should thus be used in eq. (2.3). The chirality constraints are then imposed as before, with the obvious changes.

Since the functional integrals over the left- and right-handed fermions decouple, the total effective action is the sum of the corresponding contributions. The same applies to the measure term $\mathfrak{L}_\eta$, but it can be shown that the left- and right-handed parts of the curvature $\mathfrak{F}_{\eta\zeta}$ combine to the purely left-handed expression (4.1) with the fermion representation R given by

$$R = R_L \oplus (R_R)^* . \tag{5.5}$$

In terms of this representation, the anomaly cancellation condition is that $d_R^{abc} = 0$, and the construction of the measure term thus proceeds exactly as in section 4.

It is now straightforward to add a Higgs field $\phi(x)$ that transforms according to the representation $R_L \otimes (R_R)^*$ of the gauge group. In particular, the obvious choice

$$\overline{\psi}_L(x)\,\phi(x)\,\psi_R(x) + \overline{\psi}_R(x)\,\phi(x)^\dagger\,\psi_L(x) \tag{5.6}$$

for the Yukawa interaction is manifestly gauge-invariant and perfectly acceptable. An important point to note is that the introduction of the Higgs field does not affect the measure term, because the chiral projectors (and thus the fermion integration measure) do not refer to the Higgs sector.

5.6 Calculation of electroweak processes

Lattice Feynman diagrams are relatively difficult to evaluate, but having exact gauge invariance is a definite advantage when calculating electroweak amplitudes, which may partly compensate for this. If there are only few external lines, it is quite clear that such computations are practically feasible at the one- and two-loop level.

Although they could in principle be determined algebraically following the steps taken in section 4, the vertices that derive from the measure term are not explicitly known at present. For various reasons it seems rather unlikely, however, that they will ever be needed in the cases of interest. In particular, these vertices only appear at the one-loop level and only at the fifth and higher orders of the gauge coupling. Moreover they are proportional to a positive power of the lattice spacing (a simple dimensional counting shows this) so that at one-loop order they need not be included if one is only interested in the continuum limit of the diagrams.

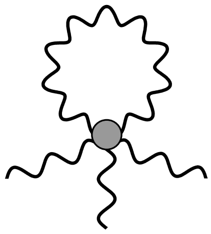

Figure 1: Feynman diagram involving the vertex $V_{\rm M}^{(5)}$ (shaded circle) that derives from the measure term. The diagram appears at two-loop order, while all other diagrams with such vertices and less than four external lines are of higher order.

At the two-loop level the situation is more complicated, but dimension counting and symmetry considerations put strong constraints on the amplitudes to which the measure term can contribute. The lowest-order (five-point) vertex, for example, is totally symmetric in the gauge group indices. Together with the lattice symmetries, Bose symmetry and locality of the vertex, this suffices to prove that the diagram shown in figure 1 vanishes in the continuum limit. One-particle irreducible diagrams at higher loop orders or with more than three external lines thus need to be considered to see a non-zero effect of the measure term.

6. Concluding remarks

Regularizations of chiral gauge theories that preserve the gauge symmetry must refer to the properties of the theory at the one-loop level, because such a regularization can only exist if the gauge anomaly cancels. For this reason simple schemes do not work out and for many years the breaking of gauge invariance thus appeared to be a necessary evil of any regularization of these theories.

In the lattice theories described in this paper the fermion integration measure has a non-trivial phase ambiguity that cannot be fixed consistently if the fermion multiplet is anomalous. The proper choice of the phase is an integral part of the definition of the lattice regularization, and the existence of the latter is thus directly linked to the presence or absence of the anomaly.

Any anomaly-free chiral gauge theory can be regularized in this way, to all orders of the perturbation expansion, but as is generally the case in lattice gauge theory, the Feynman rules tend to be rather complicated. Calculations of electroweak processes at the one- and two-loop level may nevertheless be feasible, using algebraic manipulation programs, the Reisz power counting theorem [50, 41] and a range of other tools to evaluate lattice Feynman diagrams.

An important question which has not been answered so far is whether these lattice theories are multiplicatively renormalizable. While there is little doubt that this is the case, a rigorous proof along the lines of refs. [51, 52] still needs to be given and would evidently be very welcome.

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A. Notations

A.1 Indices and Dirac matrices

Lorentz indices $\mu, \nu, \ldots$ are taken from the middle of the Greek alphabet and run from 0 to 3. The symbol $\epsilon_{\mu\nu\rho\sigma}$ denotes the totally antisymmetric tensor with $\epsilon_{0123} = 1$ and the conventions for the Dirac matrices are

$$(\gamma_\mu)^\dagger = \gamma_\mu \,, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \,, \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \,. \tag{A.1}$$

In particular, γ_5 is hermitian and $(\gamma_5)^2 = 1$.

Fermion fields on the lattice carry a Dirac index and a flavour index on which the gauge transformations act. Indices $a, b, \ldots$ from the beginning of the Latin alphabet are reserved for tensors that transform according to a tensor product of the adjoint representation of the gauge group. Unless stated otherwise the Einstein summation convention is applied.

A.2 Gauge group

Without loss the gauge group G may be assumed to be a closed subgroup of $\mathrm{U}(N)$ for some finite value of N [53]. Its Lie algebra $\mathfrak{g}$ is then a vector space of anti-hermitian matrices and there exists a basis of generators T^a $(a = 1, \ldots, n)$ such that

$$\mathrm{tr}\,\{T^aT^b\} = -\frac{1}{2}\,\delta^{ab} \,, \qquad [T^a, T^b] = f^{abc}T^c \,. \tag{A.2}$$

With these conventions, the tensor f^{abc} is real and totally antisymmetric.

JHEP06(2000)028

The representation space of the adjoint representation of $\mathfrak{g}$ is the Lie algebra itself, i.e. the elements $X = X^a T^a$ of $\mathfrak{g}$ are represented by linear transformations

$$\mathrm{Ad}\, X : \mathfrak{g} \longmapsto \mathfrak{g}\,, \qquad \mathrm{Ad}\, X \cdot Y = [X, Y] \quad \text{for all } Y \in \mathfrak{g}\,, \tag{A.3}$$

which is equivalent to

$$(\mathrm{Ad}\, X \cdot Y)^a = f^{abc} X^b Y^c \tag{A.4}$$

in terms of the components of X and Y.

A.3 Field variations

For any vector field $\eta_\mu(x)$ with values in $\mathfrak{g}$ and compact support, a first-order differential operator δ_η acting on functions of the link variables may be defined through

$$\delta_\eta f[U] = \frac{d}{dt} f[U_t] \Big|_{t=0}\,, \qquad U_t(x,\mu) = e^{ta\eta_\mu(x)} U(x,\mu)\,. \tag{A.5}$$

It is not difficult to show that $\delta_\eta f[U]$ is linear in $\eta_\mu(x)$ and that the identity

$$\delta_\eta \delta_\zeta - \delta_\zeta \delta_\eta + a\delta_{[\eta,\zeta]} = 0 \tag{A.6}$$

holds if $\eta_\mu(x)$ and $\zeta_\mu(x)$ are independent of the gauge field.

Another kind of first-order differential operator $\bar{\delta}_\eta$ acts on functions of the gauge potential according to

$$\bar{\delta}_\eta g[A] = \frac{d}{dt} g[A + t\eta] \Big|_{t=0}\,. \tag{A.7}$$

These operators commute with each other and

$$\bar{\delta}_\eta f[U] = g_0 \delta_{\bar{\eta}} f[U]\,, \qquad U(x,\mu) = e^{g_0 a A_\mu(x)}\,, \tag{A.8}$$

$$\bar{\eta}_\mu(x) = \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \left[g_0 a\, \mathrm{Ad}\, A_\mu(x) \right]^k \right\} \eta_\mu(x)\,, \tag{A.9}$$

for any function $f[U]$ of the link variables.

A.4 Lattice derivatives

The forward and backward difference operators ∂_μ and ∂_μ^* act on lattice functions according to

$$\partial_\mu f(x) = \frac{1}{a} \Big\{ f(x + a\hat{\mu}) - f(x) \Big\}\,, \tag{A.10}$$

$$\partial_\mu^* f(x) = \frac{1}{a} \Big\{ f(x) - f(x - a\hat{\mu}) \Big\}\,, \tag{A.11}$$

where $\hat{\mu}$ denotes the unit vector in direction μ. They can be made gauge-covariant by including the appropriate representation matrix of the link variables. In the case of the fermion field the covariant forward difference operator is given by

$$\nabla_\mu \psi(x) = \frac{1}{a} \Big\{ R[U(x,\mu)]\, \psi(x + a\hat{\mu}) - \psi(x) \Big\}\,, \tag{A.12}$$

and when $\omega(x)$ is a lattice field with values in $\mathfrak{g}$, the operator assumes the form

$$\nabla_\mu \omega(x) = \frac{1}{a}\Big\{ U(x,\mu)\, \omega(x+a\hat{\mu})\, U(x,\mu)^{-1} - \omega(x) \Big\}. \tag{A.13}$$

The covariant backward difference operator ∇_μ^* is defined similarly.

B. Gauge invariance of the effective action

In this appendix we prove that the effective action $S_{\rm eff}$ is gauge-invariant if conditions (a) and (b) are satisfied. Since we are only interested in the perturbative region, it suffices to show that $\delta_\eta S_{\rm eff} = 0$ for all gauge variations (2.19) and all link fields in the vicinity of the trivial field $U(x,\mu) = 1$.

We first note that the gauge-covariance of the Dirac operator,

$$\delta_\eta D = [R(\omega), D], \tag{B.1}$$

and eq. (2.15) lead to the identity

$$\delta_\eta S_{\rm eff} = \mathrm{Tr}\left\{ R(\omega)(\hat{P}_- - P_+) \right\} + i\mathfrak{L}_\eta. \tag{B.2}$$

The right-hand side of this equation is easily shown to vanish at $U(x,\mu) = 1$, using the explicit form (3.7) of the free Dirac operator in the first term and the fact that $j_\mu(x)$ cannot depend on x if the gauge field is translation-invariant. To establish the gauge invariance of the effective action, we then only need to prove that

$$\delta_\zeta \mathfrak{L}_\eta = i\,\mathrm{Tr}\left\{ R(\omega)\delta_\zeta \hat{P}_- \right\} \tag{B.3}$$

for arbitrary variations $\zeta_\mu(x)$ of the gauge field, since this implies that the right-hand side of eq. (B.2) is constant (and hence equal to zero).

Equation (B.3) is a simple consequence of the integrability condition and the gauge-covariance of the current $j_\mu(x)$. From the latter one infers that

$$\delta_\eta \mathfrak{L}_\zeta + \mathfrak{L}_{[\omega,\zeta]} = 0, \tag{B.4}$$

but the application of eq. (2.17) is a bit tricky, because $\eta_\mu(x)$ depends on the gauge field through the covariant difference operator in eq. (2.19). We may, however, get around this problem by noting that

$$\delta_\zeta \mathfrak{L}_\eta = \{\delta_\zeta \mathfrak{L}_\lambda\}_{\lambda=\eta} + \mathfrak{L}_{[\omega,\zeta]} + a\mathfrak{L}_{[\zeta,\eta]}. \tag{B.5}$$

Together with eq. (B.4), the integrability condition then yields

$$\delta_\zeta \mathfrak{L}_\eta = -i\,\mathrm{Tr}\left\{ \hat{P}_- \left[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_- \right] \right\}, \tag{B.6}$$

which reduces to eq. (B.3) when the identities

$$\delta_\eta \hat{P}_- = \left[R(\omega), \hat{P}_- \right], \qquad \hat{P}_- \delta_\zeta \hat{P}_- \hat{P}_- = 0, \tag{B.7}$$

are inserted.

C. Definition of $\mathfrak{L}_\eta^{(4)}$

A local functional $\mathfrak{L}_\eta^{(4)}$ with the required properties can be obtained from the leading-order part $\check{\mathfrak{L}}_\eta$ of the measure term by replacing the gauge potential in

$$\check{\mathfrak{L}}_\eta = \frac{1}{4!}\, a^{20} \sum_{x,\ldots,z_4} L^{(4)}(x,z_1,\ldots,z_4)^{aa_1\cdots a_4}_{\mu\mu_1\cdots\mu_4}\, \eta^a_\mu(x) A^{a_1}_{\mu_1}(z_1)\cdots A^{a_4}_{\mu_4}(z_4) \tag{C.1}$$

through the expression

$$\widehat{A}^a_\mu(x,z) = \frac{2}{a}\,\mathrm{tr}\Big\{T^a \left[1 - W(x,z)U(z,\mu)W(x,z+a\hat{\mu})^{-1}\right]\Big\}, \tag{C.2}$$

where $W(x,z)$ denotes the ordered product of the link variables from z to x along the shortest path that first goes in direction 0, then direction 1, and so on. From this definition and the invariance of $L^{(4)}$ under the adjoint action of the gauge group, it is obvious that $\mathfrak{L}_\eta^{(4)}$ is gauge-invariant. Moreover eq. (C.2) implies

$$\widehat{A}_\mu(x,z) = g_0\left\{A_\mu(z) + \partial^z_\mu \omega(x,z)\right\} + \mathrm{O}(g_0^2) \tag{C.3}$$

with $\omega(x,z)$ the "oriented line sum" of the gauge potential from z to x along the path defined above. Each term in the sum over x in eq. (C.1) is separately invariant under linearized gauge transformations, and $\mathfrak{L}_\eta^{(4)}$ thus coincides with $g_0^4\check{\mathfrak{L}}_\eta$ to leading order in the gauge coupling.

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JHEP06(2000)028

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